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Gravity Duals of Supersymmetric $SU(N) \times SU(N+M)$ Gauge Theories

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Abstract

The world volume theory on N regular and M fractional D3-branes at the conifold singularity is a non-conformal $\mathcal{N}=1$ supersymmetric $SU(N+M) \times SU(N)$ gauge theory. In previous work the Type IIB supergravity dual of this theory was constructed to leading non-trivial order in M/N: it is the $AdS_5 \times T^{1,1}$ background with NS-NS and R-R 2-form fields turned on. Far in the UV this dual description was shown to reproduce the logarithmic flow of couplings found in the field theory. In this paper we study the supersymmetric RG flow at all scales. We introduce an ansatz for the 10-d metric and other fields and show that the equations of motion may be derived in first order form from a simple superpotential. This allows us to explicitly solve for the gravity dual of the RG trajectory.

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1. Introduction

The AdS/CFT correspondence [1,2,3] is usually motivated by comparing stacks of elementary branes with corresponding gravitational backgrounds in string or M-theory. For example, the correspondence [4] between a large number N of coincident D3-branes and the 3-brane classical solution leads, after an appropriate low-energy limit is taken, to the duality between $\mathcal{N} = 4$ supersymmetric $SU(N)$ gauge theory and Type IIB strings on $AdS_5 \times S^5$ [1,2,3]. This construction gives an explicit realization of the gauge theory strings [5,6].

In order to construct the Type IIB duals of other 4-dimensional CFT's, one may place the D3-branes at appropriate conical singularities [7,8,9,10]. Then the background dual to the CFT on the D3-branes is $AdS_5 \times X_5$ where X_5 is the Einstein manifold which is the base of the cone. In addition to regular D-branes which can reside on or off the conical singularity, there are also "fractional" D-branes pinned to the singularity [11,12]. In previous work [13] the effect of such fractional branes on the dual supergravity background was considered, and it was shown how they break the conformal invariance. In this paper we continue this line of investigation, and calculate the back-reaction of the fractional branes on the gravitational background. We obtain and solve a system of first-order equations describing renormalization group (RG) flow in the gravity dual of the $\mathcal{N} = 1$ supersymmetric $SU(N) \times SU(N + M)$ gauge theory. This theory is realized on D-branes at the conifold singularity and we review it in section 2.

Our study of the gravitational RG flow builds on recent work studying such flows in gauged 5-d supergravity [14,15,16,17,18,19,20]. In particular, we will make use of the results on $\mathcal{N} = 1$ supersymmetric flows [16,21,22] which reduce second-order equations to a much simpler first-order gradient flow induced by a superpotential function (extensions of these methods to non-supersymmetric flows were given in [17,23,18,24]). In our example we start with an ansatz for the 10-d background and reduce it to a 5-d gauged supergravity coupled to scalar fields. This gives a clear geometrical meaning to the scalars fields, so that we can follow the RG evolution of the entire 10-d background. This is similar in spirit, although not in detail, to examples of RG flow found in 10-d type 0 string theory [25,26,27,28]. An interesting novel feature of the solution we find is that its existence crucially depends on the presence of the Chern-Simons term in the type IIB supergravity action.

2. RG Flow Associated with Fractional Branes on the Conifold

The conifold is a singular Calabi-Yau manifold described in terms of complex variables $w_1, \ldots, w_4$ by the equation [29] $\sum_{a=1}^4 w_a^2 = 0$. The base of this cone is $T^{1,1} = (SU(2) \times SU(2))/U(1)$ whose Einstein metric may be written down explicitly as follows [30,29],

$$ds^2_{T^{1,1}} = \frac{1}{9}\left(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i\right)^2 + \frac{1}{6}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right) \ . \tag{2.1}$$

The $\mathcal{N}=1$ superconformal field theory on N regular D3-branes placed at the singularity of the conifold has gauge group $SU(N)\times SU(N)$ and global symmetry $SU(2)\times SU(2)\times U(1)$ [9] which is a symmetry of the metric (2.1). The theory contains two chiral superfields A_i transforming as $(N,\overline{N})$ and as a doublet of the first $SU(2)$, and two chiral superfields B_k transforming as $(\overline{N},N)$ and as a doublet of the second $SU(2)$. The R-charge of all four chiral superfields is 1/2 and the theory has an exactly marginal superpotential $\mathcal{W} = \epsilon^{ij}\epsilon^{kl}\mathrm{Tr} A_i B_k A_j B_l$.

Type IIB supergravity modes on $AdS_5 \times T^{1,1}$ have been matched in some detail with operators in this gauge theory whose dimensions are of order 1 in the large N limit [31,32,33]. Topologically, $T^{1,1}$ is $S^2 \times S^3$ so that additional objects may be constructed from wrapped branes [34]. In particular, a D5-brane wrapped over the 2-cycle acts as a domain wall in AdS_5. If this domain wall is located at $r = r_w$ then, by studying the behavior of wrapped D3-branes upon crossing it, it was shown in [34] that for $r > r_w$ the gauge group changes to $SU(N+1)\times SU(N)$. This is precisely the gauge theory expected on N regular and one fractional D3-branes. Thus, a D5-brane wrapped over the 2-cycle is nothing but a fractional D3-brane placed at a definite r. The identification of a fractional D3-brane with a wrapped D5-brane is consistent with the results of [35,12,36,37].

Adding M fractional D3-branes thus produces $SU(N+M)\times SU(N)$ supersymmetric gauge theory coupled to the chiral superfields A_i and B_k. Its supergravity dual carries M units of the R-R 3-form, H_{RR}, flux through the 3-cycle of $T^{1,1}$. In [13] it was shown that this flux induces a radial variation of the integral of the NS-NS 2-form potential $\int_{S^2} B_2$. This was interpreted as the stringy dual of the logarithmic RG flow in the field theory. In the next section we set up the gravitational RG flow equations systematically, so that the back-reaction of the 3-form field strengths on other fields may be calculated. This will allow us to follow the flow far into the infrared and address the issues related to singularities.

3. The Supergravity Ansatz and the Effective Action

The type IIB supergravity equations [38] can be obtained from the action

$$S_{10} = -\frac{1}{2\kappa_{10}^2}\int d^{10}x\Big(\sqrt{-g_{10}}\big[\, R_{10} - \tfrac{1}{2}(\partial\Phi)^2 - \tfrac{1}{12}e^{-\Phi}(\partial B_2)^2 \tag{3.1}$$

$$- \tfrac{1}{2}e^{2\Phi}(\partial\mathcal{C})^2 - \tfrac{1}{12}e^{\Phi}(\partial C_2 - \mathcal{C}\partial B_2)^2 - \tfrac{1}{4\cdot 5!}F_5^2\,\big] - \tfrac{1}{2\cdot 4!\cdot(3!)^2}\epsilon_{10}C_4\partial C_2\partial B_2 + ...\Big)\ ,$$

$$(\partial B_2)_{MNK} \equiv 3\partial_{[M}B_{NK]}\ , \qquad (\partial C_4)_{MNKLP} \equiv 5\partial_{[M}C_{NKLP]}\ ,$$

$$F_5 = \partial C_4 + 5(B_2\partial C_2 - C_2\partial B_2)\ ,$$

with the additional on-shell constraint $F_5 = *F_5$ [39]. In [13] these equations were solved to leading order in M/N, and it was shown that the back-reaction on the metric and 5-form fields enters at order $(M/N)^2$. To study the back-reaction, we introduce the following ansatz. The 10-d Einstein frame metric will be chosen as a sum of the 5-d space-time metric and the internal 5-manifold metric which has the same symmetries as (2.1):

$$ds_{10}^2 = L^2\left[e^{-\frac{2}{3}(B+4C)}ds_5^2 + ds_{5'}^2\right]\ , \tag{3.2}$$

$$ds_{5'}^2 = \frac{1}{9}e^{2B}\left(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i\right)^2 + \frac{1}{6}e^{2C}\sum_{i=1}^{2}\left(d\theta_i^2 + \sin^2\theta_i d\phi_i^2\right)\ . \tag{3.3}$$

Here B, C are in general functions of the 5-d space-time coordinates and the conformal factor of the 5-d metric is chosen to preserve the Einstein frame upon compactification to 5-d. The numerical coefficients 1/9 and 1/6 (which are the same as in (2.1)) are inserted in order to have $B = C = 0$ for the $AdS_5 \times T^{1,1}$ solution. L is the scale related to the radius of AdS_5 ($L^4 \sim N$).

We set the RR scalar $\mathcal{C}$ to zero (this will be consistent with the ansatz for 3-form fields made below) and study the case where

$$ds_5^2 = du^2 + e^{2A(u)}dx_n dx_n\ , \tag{3.4}$$

and B, C and the 10-d dilaton Φ are functions of u. Following [13] we shall note that since the fractional D3-brane, i.e. the wrapped D5-brane, creates R-R 3-form flux through $T^{1,1}$, $H_{\rm RR} = dC_2$ should be proportional to the closed 3-form. This 3-form was constructed in [34],

$$H_{\rm RR} = \ P\ e^{\psi} \wedge \omega_2\ , \qquad \omega_2 \equiv \frac{1}{\sqrt{2}}(e^{\theta_1}\wedge e^{\phi_1} - e^{\theta_2}\wedge e^{\phi_2})\ . \tag{3.5}$$

Here P is a constant proportional to the integer number M of H_{RR} flux units. In the normalization where $L = 1$ which we shall use below, $P \sim M/N$ (N is fixed by the boundary conditions at $u = u_0$). We have introduced the orthonormal basis of 1-forms [34]

$$e^\psi = \frac{1}{3}(d\psi + \sum_{i=1}^{2} \cos\theta_i d\phi_i) \ , \qquad e^{\theta_i} = \frac{1}{\sqrt{6}} d\theta_i \ , \qquad e^{\phi_i} = \frac{1}{\sqrt{6}} \sin\theta_i d\phi_i \ . \tag{3.6}$$

The form of the NS-NS 2-form potential is also as in [13]

$$B_2 = \ T(u)\ \omega_2 \ , \qquad\qquad H_{\rm NS} = \ T'(u)\ du \wedge \omega_2 \ . \tag{3.7}$$

T plays the role of a scalar field in the effective 5-d supergravity theory.

The natural ansatz for the self-dual 5-form is

$$\begin{aligned} F_5 &= \mathcal{F} + *\mathcal{F} \ , \\ \mathcal{F} = K(u) \mathrm{vol}(\mathrm{T}^{1,1}) \ &= \ K(u)\ e^\psi \wedge e^{\theta_1} \wedge e^{\phi_1} \wedge e^{\theta_2} \wedge e^{\phi_2} \ , \\ *\mathcal{F} &= e^{4A - \frac{8}{3}(B+4C)}\ K\ du \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \ . \end{aligned} \tag{3.8}$$

Then $H_{\rm RR}$ in (3.5) satisfies the required equation $d * H_{\rm RR} \sim F_5 \wedge H_{\rm NS}$ since $F_5 \wedge H_{\rm NS} = 0$.

Note that our ansatz preserves the symmetry between the two 2-sphere factors in (3.3). Had we put in different warp factors for the two 2-spheres, there would still be a solution where they are equal. This is due to the special form of the ansatz for the 3-form field strengths. Thus, our ansatz is rigidly constrained and, as we will see, leads to rather simple RG equations.

The F_5 equation of motion implies a relation between the functions K and T. Indeed, $d * F_5 = dF_5 \sim H_{\rm NS} \wedge H_{\rm RR}$ implies that the scalars K and T are not independent:

$$K' = PT' \ , \quad \text{i.e.} \quad K(u) = Q + PT(u) \ . \tag{3.9}$$

For $P = 0$ the constant Q plays the role of the 5-brane charge, $Q \sim N$. For non-zero P the constant Q can be absorbed into a redefinition of the function T. This follows from the fact that $F_5 = dC_4 + 5(B_2 dC_2 - C_2 dB_2) = dC_4' + 10 B_2 dC_2$, where C_4' is related by a field redefinition to C_4,

$$C_4 = C_4' + 5 B_2 C_2 \ .$$

Since $d(dC_4') = 0$, dC_4' must contain the volume 5-form $\mathrm{vol}(\mathrm{T}^{1,1})$ part with *constant* coefficient Q.

The equation for T which follows from $d * H_{\rm NS} \sim F_5 \wedge H_{\rm RR}$ has the structure $\nabla^\mu(e^{-\Phi}\nabla_\mu T) \sim P(Q+PT)$.

In general, the dilaton will be running according to

$$\nabla^2\Phi = \frac{1}{12}(e^\Phi H_{\rm RR}^2 - e^{-\Phi} H_{\rm NS}^2) \ . \qquad (3.10)$$

However, as we shall explain below, there exists a special class of solutions for which Φ remains constant. Then $H^2_{\rm NS} = e^{2\Phi} H^2_{\rm RR}$, or $T' = Pe^{\Phi} e^{-\frac{4}{3}(B+C)}$, i.e. T can be expressed in terms of B and C.

The full set of equations for the 5-d metric function A and scalars B, C, Φ, T can be found from the following 5-d action which can be obtained from (3.1) by taking into account the solution of the F_5 equation of motion and integrating by parts

$$S_5 = -\frac{2}{\kappa_5^2}\int d^5x \, \sqrt{g_5}\left[\frac{1}{4}R_5 - \frac{1}{2}G_{ab}(\varphi)\partial\varphi^a\partial\varphi^b - V(\varphi)\right] \ . \qquad (3.11)$$

Here the scalar fields $\varphi^a = (q, f, \Phi, T)$ include the "diagonal" combinations of B and C

$$q = \frac{2}{15}(B+4C) \ , \qquad f = -\frac{1}{5}(B-C) \ , \qquad (3.12)$$

which measure the volume and the ratio of scales of the internal manifold (3.3). In the presence of a 5-d cosmological constant both turn out to be positive (mass)^2 scalars (see below). Explicitly, in (3.11)

$$G_{ab}(\varphi)\partial\varphi^a\partial\varphi^b = 15(\partial q)^2 + 10(\partial f)^2 + \frac{1}{4}(\partial\Phi)^2 + \frac{1}{4}e^{-\Phi-4f-6q}(\partial T)^2 \ , \qquad (3.13)$$

$$V(\varphi) = e^{-8q}(e^{-12f} - 6e^{-2f}) + \frac{1}{8}P^2 e^{\Phi+4f-14q} + \frac{1}{8}(Q+PT)^2 e^{-20q} \ . \qquad (3.14)$$

The kinetic term for T comes from the $e^{-\Phi}(\partial B_2)^2$ term in (3.1). The three terms in the potential have the following origin. The first combination in (3.14) comes from R_{10} in (3.1) and reflects the curvature of the internal space (3.3) present in the limit of constant "radii" e^B and e^C. The second term is the $e^\Phi(\partial C_2)^2$ evaluated on the solution (3.5). The third term corresponds to F_5^2. Note that the contribution of the Chern-Simons term in (3.1) is already effectively accounted for (it should not directly influence the 10-d gravitational part of equations of motion). As discussed above, and as apparent from the structure of the action (3.13),(3.14), for non-zero P the constant Q can be absorbed into T. This is an important feature of the system under consideration.

When written in terms of the 5-d metric function $A(u)$ and the scalars $\varphi^a(u)$ the action takes the following form

$$S_5 = -\frac{2\mathrm{Vol}_4}{\kappa_5^2} \int du\ e^{4A} \left[3A'^2 - \frac{1}{2} G_{ab}(\varphi) \varphi'^a \varphi'^b - V(\varphi) \right] . \tag{3.15}$$

The set of equations obtained by varying A, q, f, Φ, T should be supplemented by the "zero-energy" constraint

$$3A'^2 - \frac{1}{2} G_{ab}(\varphi) \varphi'^a \varphi'^b + V(\varphi) = 0 . \tag{3.16}$$

The simplest "fixed-point" solution found for $P = 0$ corresponds to the AdS_5 space: when all scalars are constant, $V = -5 + \frac{1}{8} Q^2$. In the normalization where $L = 1$ this gives the AdS_5 space of unit radius, i.e. $A = u$, for $Q = 4$.

The reader may be slightly puzzled by the origin of the rescaling that sends $N \to Q$ and $M \to P$ (recall that M is the actual number of H_{RR} quanta). If we reinstate the dependence on L and $g_s = e^\Phi$ then the kinetic term for T in (3.13) has a factor of $e^{-\Phi}/L^4$, while the N^2 and $(N+MT)^2$ terms in the potential have factors e^Φ/L^4 and $1/L^8$ respectively. Noting that the scale of the string-frame 10-d metric, which is to be held fixed, is related to the scale of the Einstein-frame metric, $L \sim N^{1/4}$, by $L_s = (g_s)^{1/4} L \sim (g_s N)^{1/4}$, we find that these three factors become $1/L_s^4$, L_s^4/N^2 and $1/N^2$ respectively. That explains why M becomes replaced by $P \sim M/N$ while N by $Q \sim 1$.

4. The First Order System and its Solution

In general, the action S_5 (3.15) generates a system of second-order differential equations. However, as was observed in [16,21], in the case of solutions preserving some amount of supersymmetry this system can be replaced by *first*-order equations in u:

$$\varphi'^a = \frac{1}{2} G^{ab} \frac{\partial W}{\partial \varphi^b} , \qquad A' = -\frac{1}{3} W(\varphi) , \tag{4.1}$$

where the superpotential W is a function of k scalars φ^a satisfying

$$V = \frac{1}{8} G^{ab} \frac{\partial W}{\partial \varphi^a} \frac{\partial W}{\partial \varphi^b} - \frac{1}{3} W^2 . \tag{4.2}$$

It is easy to check directly that (4.1),(4.2) imply the second-order equations following from (3.15),(3.16). This first-order form leads to a crucial simplification in finding the RG flow.[1]

[1] It was noted in [17,23,18] that, even in the absence of supersymmetry, any solution of the second-order system of equations that follows from (3.15) can be obtained as solution of the first-order system (4.1),(4.2). Given $V(\varphi)$, one may, in principle, solve the non-linear equation (4.2) for W. The solution is not unique, depending on k integration constants. In general, in the absence of supersymmetry, W is not guaranteed to be simple (in particular, monotonic).

Since the RG trajectory we are after is dual to $\mathcal{N}=1$ supersymmetric gauge theory, we expect the equations of motion to be simply expressible in the first order form. We have succeeded in guessing a simple superpotential W which governs these equations (for definiteness, we assume $Q,P\geq 0$):

$$W=-e^{-4q}(2e^{-6f}+3e^{4f})+\frac{1}{2}(Q+PT)e^{-10q}\ . \tag{4.3}$$

For $Q=4,\ P=0$ we get for small f and q

$$W=-3-60f^2+60q^2+\ldots\ , \tag{4.4}$$

which shows that f and q are a "good" (diagonal) choice of fields. The potential V (3.14),(4.2) has the expansion

$$V=-3+60f^2+240q^2+\ldots\ , \tag{4.5}$$

which shows that these fields have masses $m_q^2=32$ and $m_f^2=12$. The combination q in (3.12) is the standard fixed scalar degree of freedom [40] corresponding to the overall volume of the compact manifold. For the KK reduction on S^5, the superpotential for the fixed scalar was derived in [19].

Although we have not checked explicitly that the first-order flow generated by the superpotential (4.3) preserves $\mathcal{N}=1$ supersymmetry, we believe that it is the case. One indirect check of the supersymmetry is to set $P=0$ and to consider the linearized equation for q,

$$q'=4q\ .$$

Its solution is $q\sim e^{4u}$ which corresponds to adding a source for an operator of dimension 8 [2,3] (schematically, this operator has a Tr F^4 structure [40]). This describes the leading perturbation from the $AdS_5\times T^{1,1}$ background toward the metric of N D3-branes at the conifold, which is known to preserve $\mathcal{N}=1$ supersymmetry. In section 6 we exhibit the full BPS 3-brane solution, which serves as a consistency check on the first-order equations we have derived.

Note that, although V in (3.14) depends on the dilaton Φ, the superpotential W does not depend on it.[2] As a result, the dilaton remains constant along the flow! In what follows we shall set $\Phi=0$.

[2] The Φ dependence of V is reproduced in (4.2) due to the dilaton dependence of the "metric" G_{ab} entering the kinetic term of T in (3.13).

The system of equations for T, f, q that follow from (4.1),(4.3) is thus:

$$T' = Pe^{-4q+4f} \ , \tag{4.6}$$

$$f' = -\frac{3}{5}e^{-4q+4f}(1-e^{-10f}) \ , \tag{4.7}$$

$$q' = \frac{2}{15}e^{-4q+4f}(3+2e^{-10f}) - \frac{1}{6}(Q+PT)e^{-10q} \ . \tag{4.8}$$

Note that for $P=0$ this system has a stable fixed point solution $f=0,\ q=\frac{1}{6}\ln(Q/4),\ T=$ const mentioned above. It corresponds to the $AdS_5 \times T^{1,1}$ background with unit radius, and $q=0$ if $Q=4$. Therefore, we will use $Q=4$ below.

To fix the boundary conditions, we assume that above a UV cut-off scale $u=u_0$ we have the superconformal theory with $P \sim M = 0$. Physically this corresponds to inserting M fractional anti-branes at $u=u_0$. Thus we set

$$q = f = 0 \ , \qquad T = T_0 \quad \text{at} \quad u = u_0 \ , \tag{4.9}$$

and consider the solution for $u < u_0$. The constant T_0 determines the value of $g_1^{-2} - g_2^{-2}$ at the UV cut-off [9,13] (g_1, g_2 are the two gauge coupling constants). Since $\partial W/\partial f$ (the r.h.s. of (4.7)) vanishes for $f=0$, we find that $f(u)=0$ along the entire RG flow. This means that the shape of the internal manifold $T^{1,1}$ does not change, only its overall size does!

This simplifies our task to finding just two functions, $T(u)$ and $q(u)$. The first order equations for them are governed by the superpotential

$$W = -5e^{-4q} + \frac{1}{2}(Q+PT)e^{-10q} \ , \tag{4.10}$$

which is (4.3) with f set to zero. Thus, we have

$$T' = Pe^{-4q} \ , \tag{4.11}$$

$$q' = \frac{2}{3}e^{-4q} - \frac{1}{6}(Q+PT)e^{-10q} \ . \tag{4.12}$$

Introducing the variables

$$K(u) = Q + PT(u) \ , \qquad Y(u) = e^{6q(u)} \ , \tag{4.13}$$

we get from (4.11)

$$K' = P^2 e^{-4q} = P^2 Y^{-2/3} \ . \tag{4.14}$$

Using also (4.12) we find that

$$\frac{dY}{dK} = \frac{1}{P^2}(4Y - K) \ . \tag{4.15}$$

This has a general solution

$$Y = \ a_0 \ e^{4K/P^2} + \frac{K}{4} + \frac{P^2}{16} \ . \tag{4.16}$$

The constant a_0 has to be chosen to implement the UV boundary condition that $Y = 1$ when $K = K_0 = 4 + PT_0$:

$$a_0 = -\left(\frac{P^2}{16} + \frac{PT_0}{4}\right) \exp\left[-\frac{16}{P^2} - \frac{4T_0}{P}\right] \ . \tag{4.17}$$

This completely fixes the relation between T and q along the RG trajectory. In particular, for small $T - T_0$, $q = -\frac{1}{6}T_0(T - T_0) + \dots$. This is consistent with the perturbative solution of (4.6),(4.8)

$$T = T_0 + P(u - u_0) + \dots \ , \qquad q = -\frac{1}{6}PT_0(u - u_0) + \dots \ . \tag{4.18}$$

Note that $u - u_0$ translates into $\ln(\Lambda/\Lambda_0)$ in the field theory. Thus, the variation of T translates into a logarithmic flow of $g_1^{-1} - g_2^{-2}$ in the field theory confirming the result of [13]. Furthermore, we can now calculate higher order corrections to the metric and T in powers of $P \sim M/N$.

Substituting (4.16) into (4.14) we find

$$K' = P^2 \left[\frac{P^2}{16} + a_0 e^{4K/P^2} + \frac{K}{4}\right]^{-2/3} \ , \tag{4.19}$$

which in turn leads to an implicit equation for $K(u)$,

$$u_0 - u = \frac{1}{P^2} \int_K^{K_0} dz \left[\frac{P^2}{16} + a_0 e^{4z/P^2} + \frac{z}{4}\right]^{2/3} \ . \tag{4.20}$$

Using this relation and (4.16) we also have a relation between q and u.

To complete our solution, we need to find $A(u)$. The equation for the function A in (4.1) has the form

$$A' = \frac{1}{3}\left[e^{-4q+4f}(3 + 2e^{-10f}) - \frac{1}{2}(Q + PT)e^{-10q}\right] = q' + \frac{1}{P}T' + \frac{2}{3}f' \ , \tag{4.21}$$

where we have used (4.6)–(4.8) to express the exponents in terms of the derivatives. Thus *in general* A is simply a linear combination

$$A(u) = A_0 + q(u) + \frac{2}{3} f(u) + \frac{1}{P} T(u) \ . \qquad (4.22)$$

For our particular trajectory with $f = 0$, this gives A in terms of q and T. The integration constant A_0 may be shifted by a rescaling of 4-d coordinates x_n in (3.4). We can choose it so that the metric (3.4) approaches the canonical AdS_5 one, i.e. $A(u) \to u$ for $u \to u_0$. The resulting expression for the 10-d metric (3.2),(3.3),(3.4) may be written as ($L = 1$)

$$ds_{10}^2 = e^{-5q} du^2 + e^{2A-5q} dx_n dx_n + e^{3q} ds_{T^{1,1}}^2 \ , \qquad (4.23)$$

where we have used that for $f = 0$ eq. (3.12) gives $B = C = \frac{3}{2} q$. Introducing the coordinate y such that $dy = e^{-(A-A_0)} du$, we may write this metric in the form

$$ds_{10}^2 = e^{-3q+\frac{2}{P}T} (dy^2 + dx_n dx_n) + e^{3q} ds_{T^{1,1}}^2 \ . \qquad (4.24)$$

5. Solution in a Special Case

There is a simple choice of the boundary condition for T in (4.9), $T_0 = -P/4$, which leads to $a_0 = 0$. Then an explicit solution of the RG equations is straightforward: (4.16) becomes

$$Y = \frac{K}{4} + \frac{P^2}{16} \ , \qquad (5.1)$$

and we find from (4.14) that

$$Y' = \frac{P^2}{4} Y^{-2/3} \ , \qquad (5.2)$$

i.e.

$$Y(u) = e^{6q(u)} = a_1 P^{6/5} (u - u_s)^{3/5} \ , \qquad a_1 = (5/12)^{3/5} \ . \qquad (5.3)$$

Then we have

$$K = 4 + PT(u) = -\frac{1}{4} P^2 + 4 a_1 P^{6/5} (u - u_s)^{3/5} \ . \qquad (5.4)$$

Obviously, u_s is the position of the singularity where $T^{1,1}$ shrinks to vanishing size. To find the relation of u_s to the UV cut-off u_0 we note that the boundary condition (4.9) implies $Y(u_0) = 1$, i.e.

$$u_0 - u_s = \frac{12}{5P^2} \ . \qquad (5.5)$$

The effective scale factor in 5-d gauged supergravity metric (3.4) is given by (see (4.22))

$$e^{2A} \ \sim \ P^{2/5} (u - u_s)^{1/5} \exp[8 a_1 P^{-4/5} (u - u_s)^{3/5}] \ . \qquad (5.6)$$

The fact that this vanishes at $u = u_s$ seems to indicate the presence of a naked singularity in the geometry. However, recall that in the 10-d metric (4.23) the effective scale factor in front of $dx_n dx_n$ is

$$e^{2A-5q} \ \sim \ P^{-3/5} (u - u_s)^{-3/10} \exp[8 a_1 P^{-4/5} (u - u_s)^{3/5}] \ , \qquad (5.7)$$

which, in fact, blows up at u_s. Thus, in order to study the singularity structure, it is essential to know the full 10-d form of the solution.

6. More General Solutions

In this section we go beyond the 'near-horizon' approximation and construct asymptotically flat solutions of the first-order system of equations (4.1). For $P = 0$ our solution describes regular D3-branes at the conifold singularity. For $P \neq 0$ we find an interesting generalization of this solution with a logarithmically running charge.

Let us look for 10-d metric of the following 4+6 form

$$ds_{10}^2 = s^{-1/2}(r)dx_n dx_n + h^{1/2}(r)(dr^2 + r^2 ds_{T^{1,1}}^2) \ . \tag{6.1}$$

The relation of this to the notation of (4.23) is

$$s^{-1/2}(r) = e^{2A(u)-5q(u)} \ , \qquad e^{3q(u)/2} = rh^{1/4}(r) \ . \tag{6.2}$$

The new radial coordinate, r, is related to u through

$$e^{-4q(u)}du = \frac{dr}{r} \ . \tag{6.3}$$

Hence, multiplying the last two equations, we have

$$h^{1/4}(r)dr = e^{-5q(u)/2}du \ , \tag{6.4}$$

which shows that (6.1) is equivalent to (4.23).

Using (4.11) we get

$$dT = Pe^{-4q}du = Pd(\ln r) \ , \tag{6.5}$$

which has a solution

$$T(r) = \tilde{T} + P \ln r \ . \tag{6.6}$$

We already know the solution (4.16) for $q(T)$, hence we find

$$r^4 h(r) = e^{6q} = a_0 \ e^{4\tilde{Q}/P^2 + 4 \ln r} + \frac{1}{4}(\tilde{Q} + P^2 \ln r) + \frac{1}{16}P^2 \ , \tag{6.7}$$

where

$$\tilde{Q} \equiv Q + P\tilde{T} \ .$$

Thus, we have the following explicit solution for $h(r)$:

$$h(r) = b_0 + \frac{k_0 + P^2 \ln r}{4r^4} \ , \tag{6.8}$$

where

$$k_0 = \tilde{Q} + \frac{1}{4}P^2 \ , \qquad b_0 = a_0 \ e^{4\tilde{Q}/P^2} \ .$$

To solve for $s(r)$ we note that according to (6.2),(4.22)

$$s(r) = e^{-4T(r)/P+6q(r)} = \frac{1}{r^4}e^{6q(r)} = h(r) \ . \tag{6.9}$$

Remarkably, the 10-d metric thus assumes the usual "D-brane" form

$$ds_{10}^2 = h^{-1/2}(r)dx_n dx_n + h^{1/2}(r)(dr^2 + r^2 ds_{T^{1,1}}^2) \ . \tag{6.10}$$

The function $h(r)$, given in (6.8), may be written as

$$h(r) = b_0 + \frac{P^2 \ln \frac{r}{r_*}}{4r^4} \ , \tag{6.11}$$

where r_* is defined by

$$-P^2 \ln r_* = Q + P\tilde{T} + \frac{1}{4}P^2 \ . \tag{6.12}$$

The Ricci scalar for this 10-d metric turns out to have has the following simple form

$$R = -\frac{5r^{-1}h'(r) + h''(r)}{2h^{3/2}(r)} = \frac{4P^2}{\left(4b_0 r^4 + P^2 \ln \frac{r}{r_*}\right)^{3/2}} \ . \tag{6.13}$$

This makes it clear that the metric becomes singular at $r = r_s$ such that $h(r_s) = 0$.

For $P = 0$ and $b_0 > 0$ the solution (6.10) is precisely the BPS metric of D3-branes placed at the conifold singularity (note that (6.13) vanishes in this case, as it should). This shows that, at least in this case, the solution to the first order system preserves supersymmetry.

For $P \neq 0$, the solution is still asymptotically flat. Note in particular that the NS-NS field strength falls off at large r, $H_{NS} = \frac{P}{r}dr \wedge \omega_2$. A remarkable property of the solution (6.10) is that it looks like a threebrane metric whose effective charge and mass per unit volume depend on the radius logarithmically. To strengthen this interpretation, let us consider a test D3-brane oriented along the source D3-branes and placed at radial coordinate r. As in the case where the transverse 6-d CY space in (6.10) is replaced by R^6, the gravitational force on the test brane is proportional to the derivative of the metric function (6.8),

$$\mu(r) = -r^5 \frac{dh}{dr} = K(r) \ , \qquad K(r) = Q + PT(r) = \tilde{Q} + P \ln r \ . \tag{6.14}$$

For this reason $K(r)$ may be thought of as the mass per unit volume enclosed inside radius r. Note that this is different from the coefficient of the $\frac{1}{4r^4}$ term in (6.8) (i.e. $K(r)+\frac{1}{4}P^2$), but is the same as the coefficient in the 5-form field strength (3.8). This is in agreement with the expected BPS nature of this configuration. Indeed, the force on the static D3-brane probe oriented parallel to the source brane will vanish as a result of the balance between the electric force proportional to the 5-form component $\mathcal{F}$ in (3.8) and the gravitational force proportional to (6.14). The cancellation of forces is also another argument in favor of our solution preserving $\mathcal{N}=1$ supersymmetry.

To study the solution in more detail, we have to distinguish the cases where b_0 is positive, zero, or negative. The asymptotically flat region exists only if $b_0>0$. In this case, and for $P=0$, we find the AdS horizon at $r=0$. For $P\neq 0$, however, the situation is completely different because the enclosed 3-brane charge or mass density at radius r is $K(r)$. As r decreases, so does this effective 3-brane charge density. At $r=r_e>r_*$ where $K(r_e)=0$, the gravitational force changes sign, i.e. inside this radius we have antigravity. Thus, $r=r_e$ is similar to the enhancon radius found in a different setting in [41]. It is not hard to check that the metric is nonsingular at $r=r_e$ (this is obvious from (6.13)). Continuing to $r<r_e$ past r_* we eventually reach the singularity where $h(r_s)=0$, i.e. $4b_0r_s^4=P^2\ln\frac{r_*}{r_s}$. One may speculate that this singularity should be "excised" because it occurs in the region where the effective 3-brane charge and tension are negative.

The case of $b_0=a_0=0$ is the one discussed in the preceding section. The exact metric found there is simply (6.10) expressed in terms of a different radial coordinate. Again, in this case the singularity occurs at r smaller than r_e, the point where the effective 3-brane charge per unit volume vanishes. Indeed, from (5.4) we see that $K=-\frac{1}{4}P^2$ is negative at the singularity.

Finally, let us consider the case $b_0<0$. Now r cannot increase indefinitely: as r increases we find a singularity at r_+ where h vanishes. For very small r there is another curvature singularity located at $r_->r_*$. Thus, for $b_0<0$ there are two curvature singularities, and we have $r_*<r_-<r_+$. It is not clear, however, if the case $b_0<0$ is physical: if we interpret the metric (6.10) as the geometry around N regular and M fractional D3-branes placed at the conifold singularity, then this geometry is required to have the asymptotically flat region at large r.

7. Gauge Theory Interpretation and Comments

In this section we summarize some main points and further discuss the gravitational solution dual to the RG flow in the supersymmetric $SU(N+M) \times SU(N)$ gauge theory. In investigating the actual solution, we first consider the case $P \sim M/N \ll 1$. Then we see from (4.18) that near the UV cut-off u_0 both derivatives T' and q' are small, hence the gravity approximation is valid. As u decreases from u_0, $K = Q + PT$ decreases monotonically. At the value u_e given by

$$u_e = u_0 - \frac{1}{P^2} \int_0^{K_0} dz \left[\frac{P^2}{16} + a_0 \exp(4z/P^2) + \frac{z}{4} \right]^{2/3} \tag{7.1}$$

K reaches 0. Since the 5-form field strength vanishes at u_e, this location is similar to the enhancon radius found in [41]. As already mentioned above, for $u < u_e$ we find 'antigravity,' and it is plausible to assume that this region has to be excised in a full string theoretic treatment.

If we continue the effective gravity solution to $u < u_e$, we find a singularity of the metric: the value u_s where $Y = e^{6q} = 0$. Since e^{-16/P^2} is negligible for small P, we can see from (4.16) that $K(u_s) \approx -P^2/4$. Using (4.6) and (4.8) we can derive the behavior of the metric functions A, q and K. The leading behavior coincides with that found in the exact solution exhibited in section 5: near the singularity both q and A diverge to $-\infty$ as $\frac{1}{10}\ln(u - u_s)$. While e^{2A} vanishes, we note again that in the 10-dimensional metric (4.23) the conformal factor is not A, but rather

$$A - \frac{5}{2} q \approx -\frac{3}{20} \ln(u - u_s) \ , \tag{7.2}$$

i.e. the longitudinal part of the 10-d metric expands rather than contracts as we approach u_s. The 10-d metric (4.23) is nevertheless singular because of the volume of $T^{1,1}$ shrinking to zero. In particular, the 10-d Ricci scalar is $R_{10} \sim [P^2(u - u_s)]^{-3/10}$. This is why the gravity approximation near u_s has to be taken with a grain of salt: stringy corrections could alter the conclusions entirely. We find it suggestive, however, that far in the infrared the compact 5-manifold seems to be removed dynamically – this is a desirable feature for understanding the dynamics of realistic gauge theories [42]. Perhaps one day it will be possible to understand the effective 5-d string theory where the singular compact manifold is 'integrated out.'

The fate of the singularity is an interesting issue. From (4.16) we see that, for all $a_0 > -P^2/16$, the singularity is hidden behind the enhancon-type locus $K(u) = 0$ where the effective 3-brane tension vanishes. Thus, following [41] we may conjecture that such

singularities have to be excised in a string-theoretic treatment. Since negative a_0 may be unphysical, this protection of singularities may be a general phenomenon in the system we are considering.

Another issue we need to address is the fact that a change of T shifts the effective 3-brane charge. Given that T is scale dependent, it therefore appears that far in the IR the gauge group is different from that found in the UV. Let us suggest the following qualitative picture. Since Q scales as N, and P scales as M, from the point of view of the dual $SU(N+M) \times SU(N)$ gauge theory, we conjecture that N starts decreasing dynamically as the theory flows to the IR. At first, the variation of T may be interpreted as the variation of $g_1^{-2} - g_2^{-2}$ in the field theory [13]. But what happens when we reach a value of u where one of the couplings diverges? Since a shift of T corresponds to a shift of Q, the natural continuation past this infinite coupling involves the field theory with N replaced by $N - M$. Repeating this reasoning many times we seem to eventually arrive at a theory with N comparable to M or even at the theory with a single gauge group $SU(M)$. Presumably, this is the theory described by the vicinity of u_e where K is near 0. This is an intriguing conjecture, but of course we need further checks to establish it, even on a qualitative level, because of difficulties with the effective gravity approximation.

In view of the above, it appears that, even if we start with $M \ll N$, the theory dynamically drives itself into a regime where N and M are comparable. A natural question then is: why can't we start with $P \sim M/N$ of order one from the beginning. Then the problem is that, even in the UV, the flow is no longer slow and the supergravity approximation is suspect. Nevertheless, it might provide a useful qualitative picture. Even if P is of order 1, the solution typically exhibits a repulson singularity hidden behind the enhancon-type locus, similar to those found in [41].

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